

Preferences as binary relations

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Chapter 1

Definition

- A subset of ordered pairs of a set X is called a binary relation.
- Formally, R is a binary relation on X if $R \subseteq X \times X$.
- Usually we write $x R y$ if $(x, y) \in R$

Relation as Directed Graphs

Let be R a relation on a set A . A direct graph representation of relation R is $G = (A, E)$ where A is the set of nodes and E the set of direct edges where

$$(a, b) \in R \iff (a, b) \in E \text{ (an arrow from } a \text{ to } b)$$

Relation as Matrices

Let be R a relation on a set A . The matrix representation of relation R is $M_R = [m_{ab}]_{(a,b) \in R}$ where

$$\begin{cases} m_{ab} = 1 & \text{if } (a, b) \in R \\ m_{ab} = 0 & \text{if } (a, b) \notin R \end{cases}$$

Properties

A binary relation R is

- **Reflexive** if for every $x \in X$, $x R x$;
- **Irreflexive** if for every $x \in X$, $\text{not}(x R x)$
- **Complete** if for every $x, y \in X$, $x R y$ or $y R x$ (possibly both);
- **Weakly complete** if for every $x, y \in X$, $x \neq y \implies [x R y \text{ or } y R x]$ (possibly both);
- **Symmetric** if for every $x, y \in X$, $[x R y \implies y R x]$;
- **Asymmetric** if for every $x, y \in X$, $[x R y \implies \text{not}(y R x)]$;
- **Antisymmetric** if for every $x, y \in X$, $[x R y \text{ and } y R x \implies x = y]$;
- **Transitive** if for every $x, y, z \in X$, $[x R y \text{ and } y R z \implies x R z]$;
- **Negatively transitive** if for every $x, y, z \in X$,
 $[\text{not}(x R y) \text{ and } \text{not}(y R z) \implies \text{not}(x R z)]$;
- **Semi-transitive** if for every $x, y, z, t \in X$,
 $[(x R y) \text{ and } (y R z)] \implies [(x R t) \text{ or } (t R z)]$
- **Ferrers** if for every $x, y, z, t \in X$, $[(x R y) \text{ and } (z R t)] \implies [(x R t) \text{ or } (z R y)]$.

Relations P and I from R

For a binary relation R on X , we define a symmetric part I and an asymmetric part P as follows: for all $x, y \in X$

- $x I y$ if $[x R y \text{ and } y R x]$
- $x P y$ if $[x R y \text{ and } \text{not}(y R x)]$

Concatenation of two binary relations

Let be $\mathcal{R}$ and $\mathcal{R}'$ two binary relations on X . For all $x, y \in X$

$$x \mathcal{R} \bullet \mathcal{R}' y \iff \text{there exists } z \in X \text{ s.t. } [x \mathcal{R} z \text{ and } z \mathcal{R}' y]$$

Proposition

Let be $\mathcal{R}$ a binary relation on X .

- ① $\mathcal{R}$ transitive $\implies \mathcal{R} \bullet \mathcal{R} \subseteq \mathcal{R}$ (i.e. $\mathcal{R}^2 \subseteq \mathcal{R}$)
- ② $\mathcal{R}$ asymmetric $\implies \mathcal{R}$ irreflexive
- ③ $\mathcal{R}$ complete $\iff \mathcal{R}$ reflexive and weakly complete
- ④ $\mathcal{R}$ asymmetric and negative transitive $\implies \mathcal{R}$ transitive
- ⑤ $\mathcal{R}$ complete and transitive $\implies \mathcal{R}$ negative transitive

Definition

- A binary relation R on X that is reflexive, symmetric and transitive is called an **equivalence relation**.
- A binary relation R on X is a **preorder** if R is reflexive and transitive.
- A binary relation R on X is a **weak order** or a **complete preorder** if R is complete and transitive.
- A binary relation R on X is a **total order** or a **linear order** if R is complete, antisymmetric and transitive.

Exercise 1

Let be $\mathcal{B}$ a binary relation on a set $X = \{a, b, c, d, e, f\}$ defined by:

$$a \mathcal{B} a, a \mathcal{B} b, a \mathcal{B} c, a \mathcal{B} d, a \mathcal{B} e, a \mathcal{B} f$$

$$b \mathcal{B} b, b \mathcal{B} c, b \mathcal{B} d, b \mathcal{B} e, b \mathcal{B} f$$

$$c \mathcal{B} c, c \mathcal{B} d, c \mathcal{B} e, c \mathcal{B} f$$

$$d \mathcal{B} b, d \mathcal{B} c, d \mathcal{B} d, d \mathcal{B} e$$

$$e \mathcal{B} d, e \mathcal{B} e, e \mathcal{B} f$$

$$f \mathcal{B} e, f \mathcal{B} f$$

- ① Give a matrix and a graphical representation of $\mathcal{B}$
- ② Is $\mathcal{B}$ reflexive? symmetric? asymmetric? transitive? negative transitive? semi-transitive?

Exercise 2

Let $\mathcal{R}$ be a binary relation on a set A .

- $\mathcal{R}$ is said to be a *semi-order* if it is complete, semi-transitive and de Ferrers.
- The binary relation $\mathcal{R}^+$ on A defined by for all $a, b \in A$:

$$a \mathcal{R}^+ b \iff \text{for all } c \in A, \begin{cases} b \mathcal{R} c \Rightarrow a \mathcal{R} c \\ \text{and} \\ c \mathcal{R} a \Rightarrow c \mathcal{R} b \end{cases}$$

is called the *trace* of $\mathcal{R}$.

① Give the *trace* of the relation $\mathcal{B}$ above.

② Prove the following result:

Let $\mathcal{R}$ be a reflexive binary relation on A . $\mathcal{R}$ is a semi-order if and only if its trace $\mathcal{R}^+$ is complete.

How to extend a partial pre-order to a complete preorder?

By applying a topological sorting when there is **no strict cycle** in the preferences.

Idea of the numerical representation

We try to construct a binary relation $\succsim$ on X such that there exists a numerical function $f : X \rightarrow \mathbb{R}$ satisfying the property:

$$x \succsim y \iff f(x) \geq f(y)$$

In general, $\succsim$ is assumed to be a preorder.

- $x \succsim y$ means x is at least as good as y
- $\succ$ is the asymmetric part of $\succsim$
- $\sim$ is the symmetric part of $\succsim$

Proposition

Let be $\succsim$ a preorder (complete) on X representable by a function $f : X \longrightarrow \mathbb{R}$ i.e.
 $\forall x, y \in X, x \succsim y \iff f(x) \geq f(y)$

The following two properties are equivalent:

- (i) $v : X \longrightarrow \mathbb{R}$ is a function representing $\succsim$
- (ii) There exists a strictly increasing function $\varphi : f(X) \longrightarrow \mathbb{R}$ such that $v = \varphi \circ f$

Remark

f is an **ordinal scale** (See Chapter 2).

Separability

A binary relation $\succsim$ on X is said to be **separable** if there exists a countable set $Z \subseteq X$ such that, for every $x, y \in X \subseteq Z$,

if $x \succ y$ then there exists $z \in Z$ such that $x \succsim z \succsim y$

Theorem (For every X)

The following are equivalent

- (i) $\succsim$ is complete, transitive, and separable
- (ii) There is a function $u : X \rightarrow \mathbb{R}$ that representing $\succsim$

Reference

- D. Bouyssou and Ph. Vincke, Binary Relations and Preference Modeling, in “Concept and Methods for decision aiding”, Ch. 2, 2009
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